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Billiards in Finsler and Minkowski geometries

Eugene Gutkin [a], Serge Tabachnikov [b,*,1]

[a] *125 Palisades Ave., apt. 105, Santa Monica, CA 90402, USA*
[b] *Department of Mathematics, Pennsylvania State University, 218, McAllister Bldg., University Park, PA 16802, USA*

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Abstract

We begin the study of billiard dynamics in Finsler geometry. We deduce the Finsler billiard reflection law from the "least action principle", and extend the basic properties of Riemannian and Euclidean billiards to the Finsler and Minkowski settings, respectively. We prove that the Finsler billiard map is a symplectomorphism, and compute the mean free path of the Finsler billiard ball. For the planar Minkowski billiard we obtain the mirror equation, and extend the Mather's non-existence of caustics result. We establish an orbit-to-orbit duality for Minkowski billiards. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The geodesic flow of a Riemannian manifold provides a natural link between dynamics and geometry. From our viewpoint, the billiard flow is the geodesic flow on a Riemannian manifold with a boundary. In order to include important physical examples (e.g., the Boltzmann–Sinai gas), we need to allow singularities on the boundary. Thus, in a more general interpretation, billiard flows are the geodesic flows on Riemannian manifolds with corners.

* Corresponding author.
E-mail addresses: gutkine@aol.com (E. Gutkin), tabachni@math.psu.edu (S. Tabachnikov).
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Finsler geometry originated in the famous lecture of Riemann's *Uebereinige Hypothesen die der Geometrie zu Grunde liegen* but was in limbo for a long time, before gaining acceptance as a full-fledged subject. Nowadays, Finsler geometry is a very active field of research [5,8,12,13,34]. Besides being a natural generalization of Riemannian geometry, Finsler geometry has numerous applications in mathematics and physics. An example from analysis is the Teichmuller spaces endowed with the Teichmuller metric — see [26,41]. Another example is Hofer's metric of on the group of symplectomorphisms of a symplectic manifold [23,29].

From the point of view of the geometric optics, Finsler geometry describes the propagation of waves in a medium which is both anisotropic and inhomogeneous. Riemannian geometry, on the other hand, corresponds to the wave propagation in a medium, which although may be inhomogeneous, is isotropic. The Finsler geodesic flow has the same relation to Finsler geometry as the standard geodesic flow to Riemannian geometry. The dynamic and the geometric aspects of the Finsler geodesic flow continue to be actively researched [16,17].

In the present work we begin the study of Finsler billiards. They have the same relation to the Finsler geodesic flows as the conventional billiards to the (Riemannian) geodesic flows. Another way of putting it is that the Finsler billiard is the Finsler geodesic flow on a Finsler manifold with a boundary, and, more generally, on a Finsler manifold with corners. From the geometric optics point of view, the Finsler billiard describes the wave propagation in a medium, which is not only inhomogeneous and anisotropic, but also contains perfectly reflecting mirrors.

The first step in the study of Finsler billiards is the definition of the reflection law. The well-known Riemannian reflection law is local: the angle of reflection is equal to the angle of incidence. The absence of angles in Finsler geometry means that this definition has to be modified. On the other hand, the local reflection law above follows from the global principle — the "least action principle". More precisely, in Riemannian geometry a billiard orbit between a pair of points of the "billiard table" extremizes the Riemannian length in the space of all paths joining the points in question, via a reflection off of the table boundary. Likewise, the local reflection law of the Finsler billiard should be uniquely determined by the least action principle. Now the action is the Finsler length of a path.

We deduce the Finsler billiard reflection law from the "least action principle". It is just as basic as its special case — the Riemannian reflection law. Instead of the angles of incidence and reflection, which have no intrinsic meaning in the Finsler geometry, this reflection law involves the geometry of the unit sphere in the tangent space — the indicatrix. We also give a dual formulation of the Finsler reflection law in terms of the figuratrix — the unit sphere in the cotangent space. Note that in the Riemannian geometry the indicatrix and the figuratrix are naturally identified via the Euclidean metric in the tangent spaces. The *Legendre transform* extends this identification to the Finsler geometry.

The best known case of the Riemannian billiards is the Euclidean billiard: the billiard table is a domain in the Euclidean space of n dimensions. However, extensive investigations have been done only for the planar Euclidean billiard. For this model dynamical system there are beautiful theorems and tantalizing open questions [36]. The Finsler counterpart of the Euclidean space is the Minkowski space [38]. In much of the paper we specialize to the billiards in Minkowski geometry. In particular, we extend a few basic theorems on planar Euclidean billiards to the planar Minkowski billiards.

Let us now describe the contents of the paper in more detail. In Section 2 we briefly recall the basic notions of the Finsler geometry, referring the reader to the literature for more information. In Section 3 we deduce the Finsler billiard reflection law from the "least action principle". We give two equivalent formulations of the Finsler reflection law: in terms of the *indicatrix* (Lemma 3.3) and in terms of the *figuratrix* (Corollary 3.2). In the rest of this section we extend the Euclidean "string construction" (also known as the "gardener's construction") to planar Minkowski billiards. See Lemma 3.6.

In Section 4 we establish the basic facts about the Finsler billiard ball map. In particular, we give two proofs that the map preserves the natural symplectic form. See Theorem 4.3, Proposition 4.7, and Remark 4.8. As an immediate application of our technique, we compute the mean free path of the Finsler billiard ball. Our general equation (3) reduces to the classical mean free path formula when the metric is Riemannian. See Corollary 4.10. For the planar billiard the Minkowski and the Euclidean mean free path formulas essentially coincide. Compare Eqs. (5) and (7).

In Section 5 we illustrate the similarities and the differences between the Euclidean and the Minkowski billiards by three simple examples. The examples are elementary and involve the billiards in polygons. However, they reflect important themes in billiard dynamics: periodic orbits, connections with mechanics, and control of the frequency of collisions.

In Section 6 we specialize to the planar Minkowski billiard dynamics. We study the differential of the billiard map, and extend the famous Euclidean *mirror equation* of the geometric optics. See Proposition 6.1. As an application of the Minkowski mirror equation, we prove the Minkowski version of the Mather theorem. It says that convex billiard tables whose Minkowski curvature is not strictly positive do not have caustics. See Theorem 6.4.

In Section 7 we investigate a useful duality between Minkowski billiards in arbitrary dimensions. This duality allows to trade the shape of the billiard table with the metric that determines the reflection law. This phenomenon, which is peculiar to Minkowski geometry, has a continuous version, corresponding to the geodesic flows on Minkowski hypersurfaces.

2. Finsler geometry

We start with a very brief introduction to Finsler geometry (see, e.g. [2,3,5,7,8,12,13, 34,38]). There are two equivalent points of view corresponding to the Lagrangian and the Hamiltonian approaches in classical mechanics.

A Finsler metric on a manifold M is described by a smooth field of strictly convex, centrally symmetric hypersurfaces. Throughout the paper we assume that these hypersurfaces are quadratically convex (see, e.g. [3, Definition 2.1]). Each hypersurface belongs to the tangent space at a point. They are called *indicatrices*. The indicatrix consists of the Finsler unit vectors and plays the role of the unit sphere in Riemannian geometry.

Equivalently, a Finsler metric is determined by a smooth non-negative fiberwise convex Lagrangian function $L(\cdot)$ on the tangent bundle TM. The restriction of L to a tangent space T_xM gives the Finsler length of vectors in T_xM. On each space T_xM the Lagrangian L

is positive and homogeneous of degree 1. In other words, $L(x, \cdot)$ is a Banach norm in the space $T_x M$.

Given a smooth curve $\gamma : [a, b] \to M$, its length is given by

$$\mathcal{L}(\gamma) = \int_a^b L(\gamma(t), \gamma'(t))\,\mathrm{d}t.$$

The integral does not depend on the parameterization. A Finsler geodesic is an extremal of the functional $\mathcal{L}$. The Finsler geodesic flow is a flow in *TM* where the foot point of a vector in *TM* moves along the Finsler geodesic tangent to it, so that the vector remains tangent to this geodesic and preserves its norm.

Let $I_x \subset T_x M$ be the indicatrix. The figuratrix $J_x \subset T_x^* M$ in the cotangent space is defined as follows. For $u \in I_x$, consider the covector $p \in T_x^* M$ determined by Ker $p = T_u I_x$ and $p(u) = 1$. The map $D_x : I_x \to T_x^* M$ is the *Legendre transform* on the Minkowski space $T_x M$, and the figuratrix is the range of this map. Alternatively, the figuratrix is the unit sphere of the dual normed space $T_x^* M$. We will also call J_x the unit cosphere at $x \in M$. The dual Legendre transform $D_x^* : J_x \to T_x M$ is defined likewise. Let $IM \subset TM$ (resp. $JM \subset T^*M$) be the union of $I_x M, x \in M$ (resp. $J_x M, x \in M$). Then *IM* (resp. *JM*) is the Finsler unit sphere (resp. cosphere) bundle of M. Varying $x \in M$, we obtain the global Legendre transforms $D : IM \to JM$ and $D^* : JM \to IM$. The two transforms satisfy $D^* D = DD^* = 1$.

The same way as the bundle *IM* and the Lagrangian function L on *TM* correspond to one another, the field of figuratrices *JM* determines the Hamiltonian function H on T^*M. The flow of the Hamiltonian vector field sgrad H on the symplectic manifold T^*M is also called the Finsler geodesic flow. The Legendre transform identifies the two flows.

The canonical symplectic form Ω on T^*M is exact: $\Omega = \mathrm{d}\lambda$ where λ is the Liouville form on T^*M. The restriction of λ to *JM* is a contact form: $\lambda \wedge (\mathrm{d}\lambda)^{n-1} \neq 0$ everywhere on *JM*. The Hamiltonian vector field sgrad H is the Reeb field,

$$i_{\mathrm{sgrad}\,H}\,\mathrm{d}\lambda = 0, \quad (\mathrm{sgrad}\,H) = 1.$$

We will use a form of the Huygens principle of wave propagation. Let $\mathcal{F}_t$ be a propagating wave front, where t is the time variable. Fix t and consider a Finsler unit covector p, conormal to $\mathcal{F}_t$. Let p evolve under the Finsler geodesic flow for a small time ε. Then the covector p_ε is conormal to the front $\mathcal{F}_{t+\varepsilon}$.

Let M be a domain in a linear or a projective space. A Finsler metric on M is called *projective* if the geodesics are straight lines (segments). For example, a Minkowski metric on a linear space is projective. Another example is the Klein model of hyperbolic geometry. Still another example is the *Hilbert metric* inside a convex domain in the projective space — see [12]. By Hamel's theorem, the Finsler metric given by a Lagrangian $L(x, u)$ is projective if and only if the matrix L_{xu} is symmetric — see [1].

Throughout the paper we will assume that our Finsler manifold M has no conjugate points. Most of the time we will also assume that M is geodesically convex: any two points are connected by a geodesic segment.

3. Reflection law

Our definition of the reflection law for Finsler billiards is based on the variational approach: a billiard trajectory is a geodesic on a Finsler manifold with a boundary. More precisely, let M be a Finsler manifold with boundary $\partial M = N$. We will often refer to M as a *billiard table*. Let ax and xb be two geodesic segments, where the points $a, b \in M$ are in the interior of the billiard table, and $x \in N$ is on the boundary. We say that the geodesic ray xb is the billiard reflection of the geodesic ray ax if x is a critical point of the distance function $F(y) = \mathrm{dist}(ay) + \mathrm{dist}(yb)$.

Let $I = I_x$ be the indicatrix at $x \in N$, and let $u, v \in I$ be the Finsler unit vectors along the geodesic segments ax and xb, respectively. By definition, the billiard reflection is a transformation $R : I \to I$ such that $R(u) = v$. Clearly, R is the identity map on the intersection of I with the hyperplane $T_x N$ and $R(-v) = -u$. We will describe this transformation geometrically.

Fix a point $o \in M$, and consider the wave propagation from the center o. Let c be a non-singular point of the wave front $\mathcal{F}_{t_0}$ such that the geodesic segment oc of length t_0 is contained in the interior of M. The Finsler length of the segment oc extends to a smooth distance function $L(\cdot)$ in a neighborhood of c. More precisely, for every point x, sufficiently close to c, there exists a t, close to t_0, such that $x \in \mathcal{F}_t$. We set: $L(x) = t$. Let I and J be the indicatrix and the figuratrix at c, and let $D : I \to J$ be the Legendre transform. Denote by $u \in I$ the Finsler unit vector along oc. We denote by $\mathrm{d}f$ the differential of a function f.

Lemma 3.1. *At the point c we have:* $\mathrm{d}L = D(u)$.

Proof. We will use the Huygens principle, as described in Section 2. The wave front $\mathcal{F}_{t_0}$ is a level set of the Lagrangian function L. Hence $\mathrm{d}L$ annihilates the tangent plane to $\mathcal{F}_{t_0}$. By Huygens principle, $D(u)$ is a covector, conormal to this plane. Therefore $D(u)$ is proportional to $\mathrm{d}L$. It also follows from the Huygens principle that $\mathrm{d}L(u) = 1$. By definition of the Legendre transform, $D(u) \cdot (u) = 1$ as well; therefore $\mathrm{d}L = D(u)$. □

Lemma 3.1 yields the Finsler billiard reflection law.

Corollary 3.2. *The covector* $D(v) - D(u)$ *is conormal to the hyperplane* $T_x N$.

Proof. Since point $x \in N$ is a point of relative extremum of the function $\mathrm{dist}(ax) + \mathrm{dist}(xb)$, the differential of this function annihilates the tangent hyperplane $T_x N$. According to Lemma 3.1, the differential is equal to $D(v) - D(u)$. □

Now we describe the Finsler billiard reflection law in terms of the indicatrix.

Lemma 3.3. *If the affine hyperplanes* $T_u I, T_v I, T_x N \subset T_x M$ *are not parallel, then their intersection is a subspace of codimension two.*

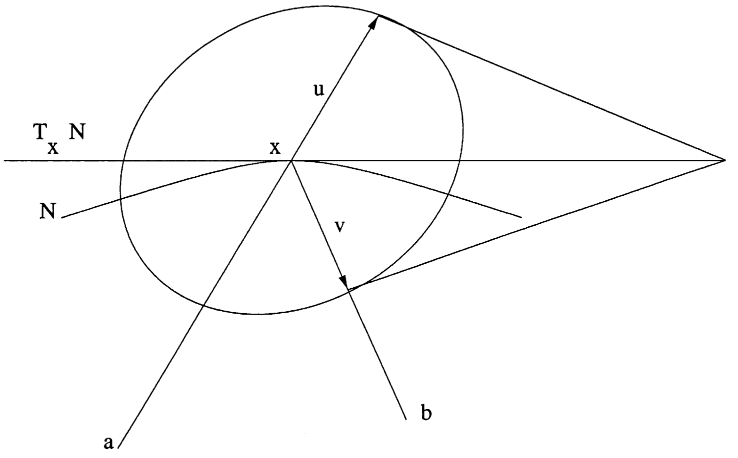

Fig. 1. Finsler billiard reflection law.

Proof. Let p be a Finsler unit covector, conormal to T_xN. If $D(v) - D(u)$ is proportional to p then the covectors $D(v)$, $D(u)$ and p are linearly dependent and span a space of atmost two dimensions. Then either $\mathrm{codim}(T_uI \cap T_vI \cap T_xN) = 2$ or T_uI, T_vI and T_xN are parallel.

Conversely, if T_uI, T_vI and T_xN are parallel, then $D(u)$ and $D(v)$ are proportional to p, hence both are conormal to T_xN. Suppose that $\mathrm{codim}(T_uI \cap T_vI \cap T_xN) = 2$. Then for every $\xi \in T_uI$ we have $D(u)(\xi) = 1$. Likewise, $D(v)(\eta) = 1$ for every $\eta \in T_vI$. Therefore $(D(v) - D(u))(\xi) = 0$ for every $\xi \in T_uI \cap T_vI = T_uI \cap T_vI \cap T_xN$. This affine subspace spans the tangent hyperplane T_XN, therefore $D(v) - D(u)$ is conormal to T_xN. □

Remark 3.4. Note that the intersection of three hyperplanes in general position has codimension three.

Fig. 1 illustrates Lemma 3.3 in two dimensions. In the Riemannian case the indicatrix is a circle, and Lemma 3.3 yields that the unit vectors u and v make equal angles with the tangent line to the boundary of the billiard table.

Example 3.5. Let a billiard table be a ball in a Finsler manifold. Then the center of the ball enjoys the same property as in the Euclidean case: a billiard trajectory passing through the center, reflects back to the center. For two points $a, b \in M$, we consider the locus of points x such that dist (xa) + dist(bx) is constant. Generically, this is a smooth hypersurface. A Finsler geodesic starting at a reflects in this hypersurface to a geodesic passing through b. This extends the familiar optical property of the Euclidean ellipse.

Let M be a geodesically convex Finsler surface with a boundary. A *caustic* is a curve, γ, in M that has the following property: if a billiard segment is tangent to γ then so is the reflected segment. In general, caustics may have singularities. In what follows we consider only convex caustics. The *string construction,* illustrated in Fig. 2, shows how

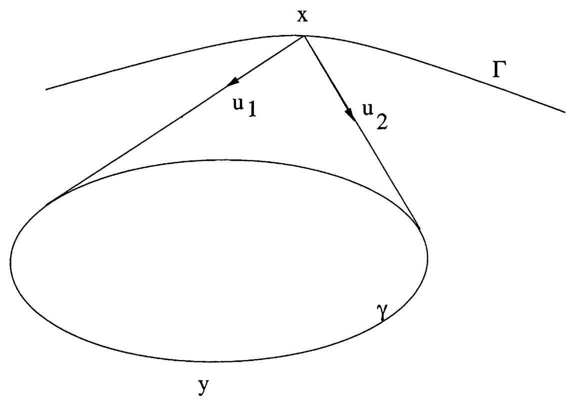

Fig. 2. String construction.

to reconstruct the billiard table from a caustic. This construction is well known for planar Euclidean billiards — see, e.g. [36].

Set $\Gamma = \partial M$. Let γ be a closed convex curve in M. For $x \in \Gamma$, let $L(x)$ be the length of the shortest curve from x to x around the "obstacle" γ. The lemma below provides a basis for the string construction.

Lemma 3.6. *The curve γ is a caustic if and only if $L(\cdot)$ is a constant function.*

Proof. Choose a point y on the "other side" of γ, and let $L_1(x)$ and $L_2(x)$ be the lengths of the two shortest curves from x to y around γ. Let u_1 and u_2 be the Finsler unit vectors at x tangent to these two curves. The argument of Lemma 3.1, shows that $-\mathrm{d}L_i = D(u_i)$, $i = 1, 2$. According to the billiard reflection law from Corollary 3.2, $R(u_1) = -u_2$ if and only if $D(u_1) + D(u_2) = -(\mathrm{d}L_1 + \mathrm{d}L_2)$ is conormal to Γ at point x. This is equivalent to Γ being a level curve of the function $L(x) = L_1(x) + L_2(x)$. □

4. Finsler billiard map and invariant symplectic structure

Let M be a billiard table, and let $N = \partial M$. It will be convenient to identify tangent and cotangent vectors via the Legendre transform.

The (Finsler) billiard map (or the billiard transformation) is the Poincaré return map for a natural cross-section of the Finsler billiard flow. Consider an oriented geodesic segment q_1q_2 in the interior of M with end-points on the boundary and transversal to it at both end-points. Let (q_1, u_1) and (q_2, u_1') be the Finsler unit tangent vectors at the end-points along this geodesic, and set $u_2 = R(u_1')$, where R is the billiard reflection at q_2. Then the billiard transformation takes (q_1, u_1) to (q_2, u_2). The phase space, Φ, of the billiard map, T, consists of inward oriented tangent vectors (covectors) with foot points on N.

We will now define a symplectic form on Φ. We will adopt the approach of Melrose. See [30,31] or [6, Chapter 3, Section 2.3], and [7, Appendix 14].

Let Ω be the canonical symplectic form on T^*M. Denote by $Y \subset T^*M$ the set of tangent covectors with foot points on N. Set $W = Y \cap JM$, and let ω be the restriction of Ω to W. Recall that we identify TM and T^*M via the Legendre transform. Denote by $\Sigma \subset W$ the subset of vectors that are tangent to N.

Lemma 4.1. *The form ω is non-degenerate on $W - \Sigma$.*

Proof. The characteristic foliation of the restriction of Ω to JM consists of the trajectories of the geodesic flow. But $W - \Sigma$ is transversal to the leaves. □

We denote by K the characteristic foliation of the restriction of Ω to Y.

Lemma 4.2. *Let $(q, p) \in Y$ and let n be a covector at q, conormal to N. Then the leaf of K containing (q, p) consists of covectors $(q, p + tn)$, $t \in \mathbb{R}$.*

Proof. We will use the standard notational conventions. If p, q and n are vectors with the same number of components, then

$$\mathrm{d}p \wedge \mathrm{d}q = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 + \cdots, \qquad \frac{n\partial}{\partial p} = \frac{n_1\partial}{\partial p_1} + \frac{n_2\partial}{\partial p_2} + \cdots.$$

In our notation, $\Omega = \mathrm{d}p \wedge \mathrm{d}q$. Consider the line $(q, p + tn)$, $t \in \mathbb{R}$ which lies in the fiber of the cotangent bundle T^*M over the point q. The vector $\xi = n\partial/\partial p$ is tangent to this line. Then $i_\xi \Omega = n\,\mathrm{d}q$, and we will show that this 1-form vanishes on Y. Indeed, the tangent space $T_{(q,p)}Y$ is the direct sum of the "horizontal" subspace T_qN and the "vertical" subspace T_q^*M. The 1-form $n\,\mathrm{d}q$ vanishes on the latter, and if $w \in T_qN$ then $(n\,\mathrm{d}q)(w) = n \cdot w = 0$ since n is a conormal vector to N. Thus ξ belongs to the characteristic direction in Y. □

Now we will prove the main result of this section.

Theorem 4.3. *The form ω on Φ is symplectic, and is invariant under the billiard map.*

Proof. Since $\Phi \subset W - \Sigma$, Lemma 4.1 implies the first claim. To prove the second claim, we will use the notation introduced in the beginning of the section. Let $\mathcal{U}_1 \subset \Phi$ be a sufficiently small neighborhood of (q_1, u_1). Denote by $\mathcal{U} \subset W$ (resp. $\mathcal{U}_2 \subset \Phi$) the corresponding neighborhood of (q_1, u_1') (resp. (q_2, u_2)). Then $T(q_1, u_1) = (q_2, u_2)$ and $T(\mathcal{U}_1) = \mathcal{U}_2$. Since the subsets $\mathcal{U}_1$ and $\mathcal{U}$ of JM are transversal to the characteristic foliation of the restriction of Ω to JM, and the intersection with its leaves is a diffeomorphism, $T' : \mathcal{U}_1 \to \mathcal{U}$, we obtain that T' is a symplectomorphism.

The same argument, applied to the subsets $\mathcal{U}$ and $\mathcal{U}_2$ of Y implies that the natural correspondence between $\mathcal{U}$ and $\mathcal{U}_2$ is a symplectomorphism. By Lemma 4.2, this symplectomorphism is the Finsler billiard reflection map R. Since $T = RT'$, the claim follows. □

Remark 4.4. Let M with the boundary N be a compact Finsler billiard table. Suppose that for any pair of points in N there is a unique geodesic segment in $\mathrm{int}(M)$, joining them. Then the space $\mathcal{R}$ of oriented geodesic segments in M is naturally identified with Φ. Under this identification, the form ω of Theorem 4.3 coincides with the canonical symplectic structure on the space of oriented geodesics. The billiard transformation becomes a symplectomorphism of $\mathcal{R}$.

Remark 4.5. Lemma 4.2 identifies the space of leaves of the characteristic foliation, K, of $\Omega|_Y$ with T^*N. The projection $\pi : W - \Sigma \to T^*N$ along the leaves of K is a two-to-one map, and its range is the open unit coball bundle B^*N. Let ω_0 be the canonical symplectic form on T^*N. It follows from Theorem 4.3 and Lemma 4.2 that $\pi^*(\omega_0) = \omega$. In the Riemannian case, π is the orthogonal projection on T^*N.

Remark 4.6. Let M be a geodesically convex compact Finsler domain of two dimensions. Then Φ is a topological cylinder. The leaves of its "vertical" foliation consist of tangent vectors with the same foot point. The billiard map satisfies the twist condition, just as in the Euclidean case (see [24]).

There is another approach to the invariant symplectic structure for the billiard map, namely via a *generating function*. See [24] for the classical case. Let M satisfy the assumptions of Remark 4.6, except that its dimension can be arbitrary. We identify the phase space Φ with the set of geodesic segments q_1q_2 in the interior of M, and use the notation developed in the proof of Theorem 4.3. Let p_1, p_1' and p_2 be the dual covectors of u_1, u_1' and u_2, respectively. Denote by $L(q_1, q_2)$ the Finsler length of q_1q_2, and let π be the projection on T^*N — see Remark 4.5.

Proposition 4.7. *In the notation above, we have:*

$$\frac{\partial L(q_1, q_2)}{\partial q_1} = -\pi(p_1), \qquad \frac{\partial L(q_1, q_2)}{\partial q_2} = \pi(p_2).$$

Proof. If q_1, q_2 were not constrained, then, by Lemma 3.1, $\partial L(q_1, q_2)/\partial q_1 = -p_1$, $\partial L(q_1, q_2)/\partial q_2 = p_2$. Taking the constraints into account

$$\frac{\partial L(q_1, q_2)}{\partial q_1} = -\pi(p_1), \qquad \frac{\partial L(q_1, q_2)}{\partial q_2} = \pi(p_1').$$

But, by Corollary 3.2, $\pi(p_1') = \pi(p_2)$. □

Remark 4.8. Proposition 4.7 provides another proof that the billiard map is symplectic. As in Remark 4.5, we identify Φ with the open unit coball bundle B^*N. Proposition 4.7 implies that

$$\pi(p_2)\,\mathrm{d}q_2 - \pi(p_1)\,\mathrm{d}q_1 = \mathrm{d}L(q_1, q_2). \tag{1}$$

Therefore

$$\pi(p_2) \wedge \mathrm{d}q_2 = \mathrm{d}\pi(p_1) \wedge \mathrm{d}q_1. \tag{2}$$

Hence $T^*(\omega_0) = \omega_0$.

To illustrate the usefulness of the preceding formalism, we will obtain a few formulas relating the Finsler geometry and the billiard statistics. For the Euclidean counterpart of our results see the classical book [35]. We will compute the mean free path of the Finsler billiard ball. More precisely, let M^n be a Finsler billiard table and let N be its boundary. The symplectic structure ω on the phase space Φ induces the invariant volume form,

$$\nu = \frac{1}{(n-1)!}\omega \wedge \cdots \wedge \omega \quad (n-1\,\text{times}).$$

The length $L(q_1, q_2)$ of a segment of a billiard orbit has the physical meaning of the free path of the ball between consecutive collisions with the boundary. The mean value $\bar{L}$ of this function on Φ is an important physical (and dynamical) characteristic of the billiard ball problem.

Let Ψ be the phase space of the billiard flow in M. Recall that λ is the Liouville 1-form on T^*M. Since Ψ is naturally identified with the hypersurface $JM \subset T^*M$, the contact volume form on Ψ is given by

$$\mu = \frac{1}{(n-1)!}\lambda \wedge \mathrm{d}\lambda \wedge \cdots \wedge \mathrm{d}\lambda \quad (n-1\,\text{times}).$$

The measures μ and ν on the phase spaces Ψ and Φ are invariant under the billiard flow and the billiard map, respectively. Let $\mathrm{vol}(\Psi)$ and $\mathrm{vol}(\Phi)$ be the corresponding volumes.

Proposition 4.9. *The mean free path of the Finsler billiard ball is equal to the volume ratio:*

$$\bar{L}_{\mathrm{Fin}} = \frac{\mathrm{vol}(\Psi)}{\mathrm{vol}(\Phi)}. \tag{3}$$

Proof. We have the inclusion $\Phi \subset \Psi$. The space Φ is a cross-section of the Finsler billiard flow, and the billiard transformation is the Poincaré return map. The invariant volume forms are related by $\nu = i_{\mathrm{sgrad}\,H}\mu$. The claim follows by an application of Fubini's theorem. □

Specializing Eq. (3), we will reproduce some classical formulas. Let M^n be a Riemannian manifold with corners, and let $N^{n-1} = \partial M$. Denote by $\mathrm{vol}(M)$ and $\mathrm{vol}(N)$ the corresponding Riemannian volumes. Let $\mathrm{vol}_{\mathrm{E}}(B^n)$ and $\mathrm{vol}_{\mathrm{E}}(S^n)$ be the Riemannian volumes of the Euclidean unit ball and the Euclidean unit sphere of n-dimensions.

Corollary 4.10. *The mean free path for the billiard ball problem in a Riemannian manifold with corners, of n dimensions, is given by*

$$\bar{L} = \frac{\mathrm{vol}_{\mathrm{E}}(S^{n-1})}{\mathrm{vol}_{\mathrm{E}}(B^{n-1})}\frac{\mathrm{vol}(M)}{\mathrm{vol}(N)}. \tag{4}$$

Proof. The symplectic and contact volumes in Proposition 4.9 have simple expressions in terms of the Riemannian volumes:

$$\mathrm{vol}(\Psi) = \mathrm{vol}_{\mathrm{E}}(S^{n-1})\mathrm{vol}(M), \qquad \mathrm{vol}(\Phi) = \mathrm{vol}_{\mathrm{E}}(B^{n-1})\mathrm{vol}(N).$$

Eq. (3) implies the claim. □

A particularly simple case of Corollary 4.10 is when the billiard table is a planar domain, $\Omega \subset \mathbb{R}^2$. Then Eq. (4) yields the well-known formula for the mean free path in Ω (see, e.g. [35]):

$$\bar{L}(\Omega) = \frac{\pi \times \mathrm{Area}(\Omega)}{\mathrm{Perimeter}(\Omega)}. \tag{5}$$

The notion of Euclidean volume has several counterparts in Minkowski geometry. Let X be a k-dimensional Finsler manifold. The Holmes–Thompson volume, $\mathrm{vol}_{\mathrm{HT}}(X)$, is the symplectic volume of the unit codisk bundle of X divided by the volume of the Euclidean k-dimensional unit ball. See, e.g. [3, Definition 3.4]. The formula below is proved the same way as Corollary 4.10, with the Holmes–Thompson volume instead of the Riemannian.

Corollary 4.11. *Let $M \subset V$ be a Minkowski billiard table of n dimensions and let N be the boundary of M. The mean free path of the Minkowski billiard in M is given by*

$$\bar{L}_{\mathrm{Min}} = \frac{\mathrm{vol}_{\mathrm{E}}(S^{n-1})}{\mathrm{vol}_{\mathrm{E}}(B^{n-1})}\frac{\mathrm{vol}_{\mathrm{HT}}(M)}{\mathrm{vol}_{\mathrm{HT}}(N)}. \tag{6}$$

Let us now specialize to the planar Minkowski billiard. If Ω is a planar Minkowski domain, then the Holmes–Thompson volume of its boundary coincides with the Minkowski perimeter. We denote it by $\mathrm{Perimeter}_{\mathrm{Min}}(\Omega)$. Corollary 4.11 yields the Minkowski counterpart of the formula (5) for the mean free path in Ω:

$$\bar{L}_{\mathrm{Min}}(\Omega) = \frac{\pi \times \mathrm{Area}_{\mathrm{HT}}(\Omega)}{\mathrm{Perimeter}_{\mathrm{Min}}(\Omega)}. \tag{7}$$

5. Three elementary examples

5.1. *Fagnano orbits in Minkowski triangles*

Periodic billiard orbits in Euclidean polygons is a fascinating subject, where even the most basic questions remain open. See [18,27] and the bibliography there. In particular, it is not known if every Euclidean triangle has a periodic orbit. Every acute triangle does: it is the so-called Fagnano orbit — see [15] or [19]. This 3-periodic orbit is realized by the inscribed triangle of the minimal perimeter. Denote the triangular billiard table by *ABC*. Then the vertices of the Fagnano triangle *PQR* are the base points of the altitudes of *ABC*. Note that the Fagnano orbit degenerates into a singular one if *ABC* is a right triangle. In the parameter space of Euclidean triangles the set of right triangles is the boundary between the subsets of acute and obtuse triangles.

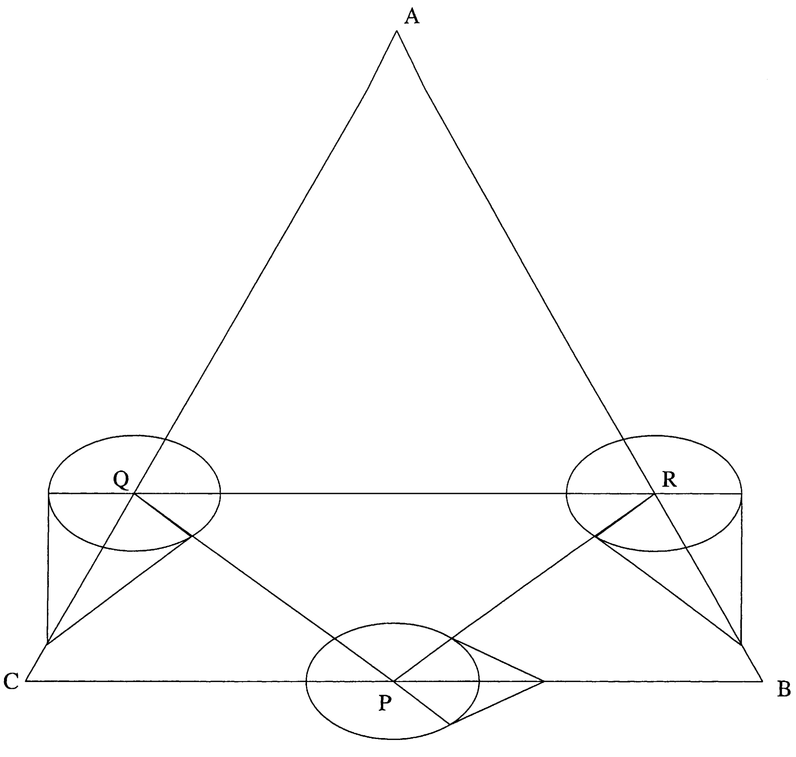

Fig. 3. Constructing the triangle from a Fagnano trajectory.

We will investigate the Fagnano orbit in Minkowski triangles, i.e., triangles in a Minkowski plane. Let I be the indicatrix, let ABC be a Minkowski triangle. If the Fagnano triangle, PQR, exists, we say that ABC is an *acute Minkowski triangle*. We say that ABC is a *right Minkowski triangle* if PQR degenerates into a segment with one of its end-points at a vertex of ABC, while the other end-point belongs to the interior of the opposite side of ABC. Denote by $\mathcal{A} = \mathcal{A}(I)$ (resp. $\mathcal{R} = \mathcal{R}(I)$) the set of acute (resp. right) Minkowski triangles. Let $\mathcal{T}$ be the set of all Minkowski triangles.

Proposition 5.1. *The geometric construction shown in Fig.* 3 *associates with any triangle $PQR \in \mathcal{T}$ the unique $ABC \in \mathcal{A}$ such that PQR is the Fagnano orbit in ABC. Regarding segments as degenerate triangles, and applying the same construction to them, we obtain all right Minkowski triangles — see Fig.* 4.

Proof. Lemma 3.3 implies the construction of Fig. 3. The first claim follows. If PQR degenerates into a segment, we obtain, as a limit of Fig. 3, the construction shown in Fig. 4. □

Note that if I is a circle, we recover from Figs. 3 and 4 the well-known Euclidean constructions.

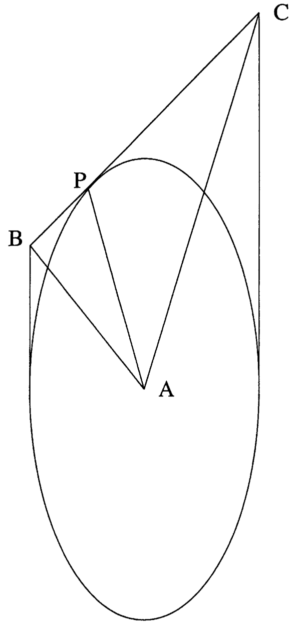

Fig. 4. Triangles with a degenerate Fagnano trajectory.

5.2. Point masses in one dimension

One of the most basic physical problems concerns the motion of the gas of one-dimensional point masses. The particle of mass m moving with velocity v contributes $\frac{1}{2}mv^2$ to the energy of the system. The gas particles move freely, and interact by perfectly elastic collisions. Using the standard conservation laws (energy and linear momentum), this physical system reduces to a Euclidean billiard. In particular, the system of two elastic point masses on the line (resp. half-line, resp. interval) is isomorphic to the Euclidean billiard in the half-plane (resp. a wedge, resp. a right triangle) (see [14,21,25,36]).

The main reason that these physical systems reduce to a Euclidean billiard is that the kinetic energy is quadratic in the velocities. In the example below we investigate the system of elastic point masses in one dimension, with a non-quadratic dependence of the energy on the velocities.

For simplicity of exposition, we consider only the case of two particles. Let m_1 and m_2 be the masses. Denote the velocities by u_1 and u_2. We fix $\lambda > 1$. The corresponding λ-*energy* is

$$E_\lambda = \frac{1}{\lambda}(m_1|u_1|^\lambda + m_2|u_2|^\lambda).$$

The λ-*momentum* is given by

$$P_\lambda = \frac{1}{\lambda - 1}(m_1|u_1|^{\lambda-1} + m_2|u_2|^{\lambda-1}).$$

Proposition 5.2. *The system of two perfectly elastic point masses on the line, whose interaction preserves the λ-energy and the λ-momentum is isomorphic to the Minkowski billiard in a half-plane, where the metric is given by*

$$L(v_1, v_2) = (v_1^\lambda + v_2^\lambda)^{1/\lambda}. \quad (8)$$

Proof. Let x_1, x_2 be the coordinates of the particles. The configuration space of our system is the half-plane $x_1 \le x_2$. The line $x_1 = x_2$ is the collision locus. Set $y_i = m_i^{1/\lambda} x_i$, $i = 1, 2$. These are the new coordinates in the configuration space. The corresponding new velocities are given by $v_i = m_i^{1/\lambda} u_i$, $i = 1, 2$. The new configuration space is the half-plane

$$m_1^{-1/\lambda} y_1 \le m_2^{-1/\lambda} y_2.$$

The collision locus

$$m_1^{-1/\lambda} y_1 = m_2^{-1/\lambda} y_2$$

is tangent to the vector $\tau = (m_1^{1/\lambda}, m_2^{1/\lambda})$.

In these coordinates the λ-momentum and λ-energy become

$$P_\lambda = m_1^{1/\lambda} |v_1|^{\lambda-1} + m_2^{1/\lambda} |v_2|^{\lambda-1}, \qquad E_\lambda = |v_1|^\lambda + |v_2|^\lambda.$$

Thus the energy is the λth power of the Minkowski Lagrangian (8). The Legendre transform $D(v_1, v_2) = (p_1, p_2) = \vec{p}$ is given by $p_i = v_i^{\lambda-1}$, $i = 1, 2$. For the λ-momentum we have

$$P_\lambda = m_1^{1/\lambda} p_1 + m_2^{1/\lambda} p_2 = \vec{p} \cdot \tau.$$

The motion of the two particles before (resp. after) a collision corresponds to the motion of the configuration point $\vec{y} = (y_1, y_2)$ with a constant velocity $\vec{v} = (v_1, v_2)$ (resp. $\vec{v}' = (v_1', v_2')$). The transformation $R : \vec{v} \to \vec{v}'$ is determined by the conservation laws E_λ and P_λ. Let $\vec{p}$ and $\vec{p}'$ be the corresponding covectors. The conservation of E_λ implies that $\vec{p}$ and $\vec{p}'$ have the same Minkowski length, while the conservation of P_λ means that $(\vec{p} - \vec{p}') \cdot \tau = 0$. By Corollary 3.2, $R : \vec{v} \to \vec{v}'$ is the Minkowski reflection law. □

5.3. Minkowski billiard in a wedge

A planar wedge is the domain bounded by two intersecting half-lines. The billiard in a wedge naturally arises in a number of problems, in particular, in polygonal billiards. The main property of the billiard orbits in a Euclidean wedge is that the total number of reflections is uniformly bounded. If α is the angle of the wedge, then any billiard orbit in it makes at most $\lceil \pi/\alpha \rceil$ collisions with the boundary. We will obtain an analog of this property for Minkowski billiards.

Proposition 5.3. *For every Minkowski wedge C there is a uniform upper bound on the number of bounces of Minkowski billiard orbits in C.*

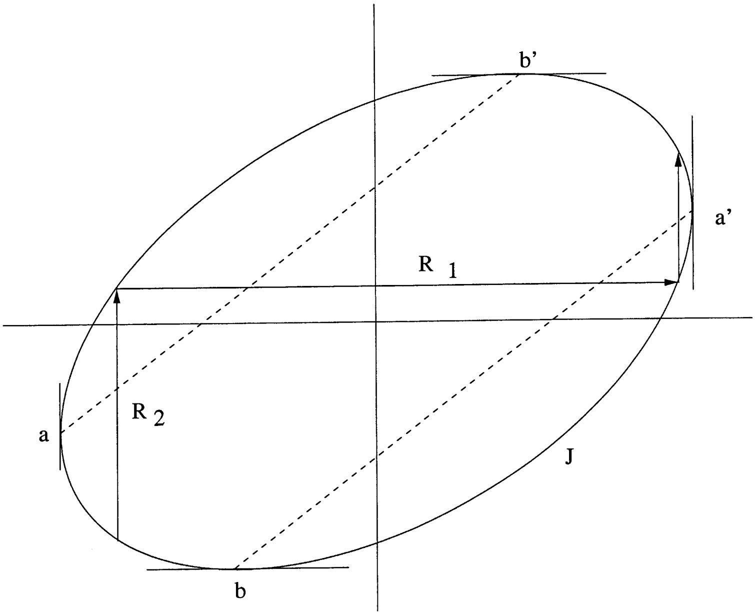

Fig. 5. Minkowski billiard reflections in a wedge.

Proof. Let J be the figuratrix. Denote by l_1, l_2 the two sides of C, and let $p_1, p_2 \in J$ be the inward conormals. Without loss of generality, we assume that p_1 and p_2 have the horizontal and the vertical directions — see Fig. 5.

The billiard reflections $R_i : J \to J, i = 1, 2$ in the respective sides of C have a geometric description. For $x \in J$ we draw the half-line from x in the direction of p_i. If it intersects J then the intersection point is $R_i(x)$. Let $a, a', b, b' \in J$ be the points with the vertical or horizontal tangent lines.

Each of the transformations R_i is defined on a proper subarc of the figuratrix. The arc $a'b' \subset J$ is the set of covectors for which neither reflection is defined. They are the Legendre transforms of the exit directions from the wedge. The opposite arc $ab = -a'b'$ corresponds to the entrance directions. For convenience, we extend R_1 (resp. R_2) to the arc $b'b$ (resp. aa') as the identity map.

A trajectory entering the wedge C in a direction x hits one of its sides, say l_1. Upon reflection, the direction becomes R_1x. Then the trajectory hits the side l_2, and the direction becomes R_2R_1x. The process continues until in this sequence of directions we obtain an exit direction. From that time on, the trajectory continues straight, without encountering the sides of C.

Hence, it suffices to prove that there exists a number $n \geq 1$ with the following property: for every $x \in ab \subset J$ both $(R_1R_2)^n(x)$ and $(R_2R_1)^n(x)$ belong to the arc $a'b'$. Consider the segments $[a, b']$ and $[ba']$ that are symmetric with respect to the origin. Acting by diagonal matrices, if necessary, we can assume that $[a, b']$ has length one and the slope one. Denote by δ be the distance between the parallel segments $[ab']$ and $[ba']$. Note that δ depends only on C and J.

Suppose that $x \in ab'$ and $R_1(x) \in ba'$. Then the orthogonal projection of the segment $[x, R_1(x)]$ on the line ab' has length at least δ. If $x \in ba'$, the same holds for the segment $[x, R_2(x)]$. Hence we may take for n the smallest integer which is greater than $2 + 1/\delta$. □

6. Mirror equation for Minkowski billiards

We begin by recalling the classical *mirror equation* of the geometric optics. Let Γ be a curve in the Euclidean plane which serves as a mirror. Let an infinitesimal beam of light emanate from a point A, and reflect in the mirror at a point X. Let the reflected beam focus at B. The mirror equation is a relation between the focusing distances $a = |AX|$ and $b = |BX|$, the angle θ of incidence of the beam (equal to the angle of reflection), and the curvature K of Γ at the point X,

$$\frac{1}{a} + \frac{1}{b} = \frac{2K}{\sin\theta}. \quad (9)$$

Fig. 6 illustrates this equation. Note that a and b in (9) are signed distances. The signs depend on the side of the mirror where the focusing points are located. Formula (9) has numerous applications in the Euclidean billiard dynamics. See [9,20,42,43]. In this section we will extend the mirror equation to Minkowski billiards.

We recall a few basic concepts of the differential geometry in a Minkowski plane. Let X be a point of a C^2 smooth, oriented curve Γ. Then there is a unique $\rho \in (0, \infty]$ such that the indicatrix I, scaled by ρ, is tangent of order two to Γ at X. It is called the *osculating indicatrix*. The number ρ is the *Minkowski radius of curvature* of Γ at X, and $|K| = 1/\rho$ is the absolute value of the *Minkowski curvature* (see [2,3,37]). Let $\vec{t}(X)$ be the tangent vector at X, and let $\vec{n}(X)$ be the vector going from X to the center of the osculating indicatrix,

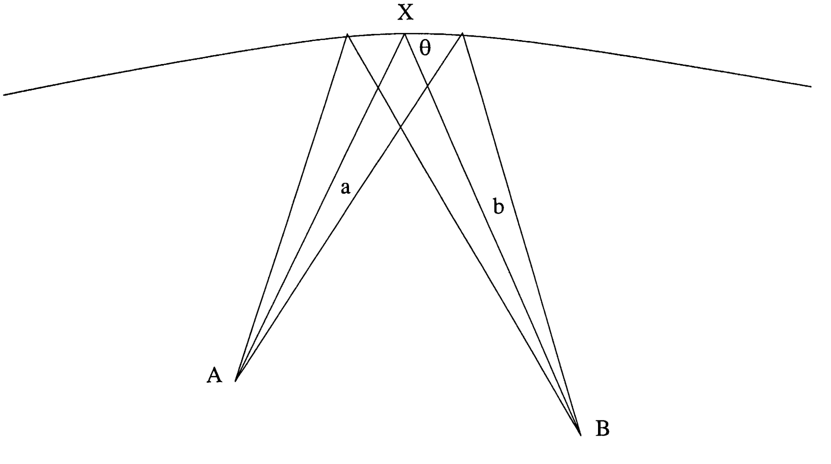

Fig. 6. The geometry of the mirror equation.

i.e., the curvature vector. Then the sign of the Minkowski curvature coincides with the orientation of the frame $(\vec{t}(X), \vec{n}(X))$.

We denote by $L(\cdot)$ the Minkowski norm of a vector. If X and Y are points in the Minkowski plane, we use the notation $L(X-Y)$ for the Minkowski distance. Choose a parameterization $\Gamma(t)$, so that $X = \Gamma(t_0)$, and set

$$F(t) = L(\Gamma(t) - A) + L(\Gamma(t) - B). \tag{10}$$

The relation $F'(t_0) = 0$ corresponds to the Minkowski reflection law. If A is a point on the incoming ray, interpreted as the focusing point of the incoming beam, $F'(t_0) = 0$ holds for any choice of B in Eq. (10). But the relation $F''(t_0) = 0$ will hold for a unique B in Eq. (10). This B is the focusing point of the outgoing beam. Thus the vanishing of the second derivative for the function defined by Eq. (10) is, in the nutshell, the Minkowski mirror equation. In order to obtain the Minkowski mirror equation in a form analogous to the Euclidean formula (9), we will expand the function F of Eq. (10) up to the second order.

We introduce the notation that will be used in the Minkowski mirror equation. Let u and v be the Minkowski unit vectors along AX and BX, and set $a = L(A-X)$, $b = L(B-X)$. The relation $F'(t_0) = 0$ yields

$$(\mathrm{d}L(u) + \mathrm{d}L(v))(\Gamma'(t_0)) = 0.$$

This is, of course, the Minkowski reflection law. We regard the second differential $\mathrm{d}^2L(\cdot)$ as a function on the Minkowski plane whose values are the bilinear forms (on the same plane). Since $L(\cdot)$ is homogeneous of degree 1, $\mathrm{d}^2L(\cdot)$ is homogeneous of degree -1. We have

$$\frac{\mathrm{d}^2L(u)(\Gamma'(t_0), \Gamma'(t_0))}{a} + \frac{\mathrm{d}^2L(u)(\Gamma'(t_0), \Gamma'(t_0))}{b} + (\mathrm{d}L(u) + \mathrm{d}L(v))(\Gamma''(t_0))=0. \tag{11}$$

Eq. (11) does not depend on the parameterization of Γ. Moreover, it does not change if we replace Γ by any other curve that is tangent to it up to the second order at X. We use the osculating indicatrix for this purpose. Denote by I' and I'' the first and the second derivative vectors, respectively (with respect to some parameterization) of the indicatrix at the osculating point. Substituting these quantities into (11), we obtain the Minkowski mirror equation.

Proposition 6.1. *Let Γ be a curve, serving as a mirror in the Minkowski plane. Let I be the indicatrix of the metric. Suppose that an infinitesimal beam of light rays emanates from a point A, reflects in the mirror at a point X, and refocuses at B. Let K be the Minkowski curvature of Γ at X. Then*

$$\frac{\mathrm{d}^2L(u)(I', I')}{a} + \frac{\mathrm{d}^2L(v)(I', I')}{b} + K[\mathrm{d}L(u) + \mathrm{d}L(v)](I'') = 0. \tag{12}$$

For some applications it is convenient to reformulate the formula above using the square of the Minkowski norm. Set $Q(\cdot) = \frac{1}{2}L(\cdot)^2$. We will use the same interpretations for $\mathrm{d}Q$

and $\mathrm{d}^2 Q$ as for the corresponding differentials of L. Eq. (12) becomes

$$\frac{\mathrm{d}^2 Q(u)(I', I') - (\mathrm{d}Q(u)(I'))^2}{a} + \frac{\mathrm{d}^2 Q(v)(I', I') - (\mathrm{d}Q(v)(I'))^2}{b} + K[\mathrm{d}Q(u) + \mathrm{d}Q(v)](I'') = 0. \tag{13}$$

Remark 6.2. Let the metric be Euclidean. Then $I(t) = (\cos t, \sin t)$, the first differential satisfies $\mathrm{d}Q(u) = u$, and the second differential $\mathrm{d}^2 Q$ is the identity. Let θ be the angle of incidence of the infinitesimal beam. Then

$$\mathrm{d}Q(u)(I') = \mathrm{d}Q(v)(I') = \cos\theta, \qquad \mathrm{d}Q(u)(I'') = \mathrm{d}Q(v)(I'') = -\sin\theta.$$

Substituting this into (13), we obtain the classical mirror equation (9).

By definition, an *invariant circle* of a twist map is a homotopically non-trivial topological circle in the phase cylinder, which is invariant under the map. We refer to [24] for more information on the subject. Among various results on the non-existence of invariant circles, the theorem of Mather's [28] stands out. Mather considers the Euclidean billiard inside a convex curve of class C^2 — the Birkhoff billiard table. He proves that if the curvature of the table has a zero, then the billiard map has no invariant circles. This is to be contrasted with the existence theorems for invariant circles, if the curvature is strictly positive [24].

As an application of Eq. (12), we will extend Mather's theorem to Birkhoff billiards in a Minkowski plane. Let N be the boundary of a Birkhoff billiard table M. An invariant circle, C, for the billiard map is given by a Lipschitz mapping of the standard circle into the space, Φ, of oriented lines intersecting the interior of M. This follows from the Birkhoff theorem — see [10,11,22,24]. The envelope, γ, of this family of lines is the *billiard caustic* corresponding to the circle C.

Heuristically, γ is a caustic for the mirror N means that any light ray, tangent to γ at some point, remains tangent to it (at another point) after the reflection in N. Caustics tend to have singularities. Hence, to make the statement above precise, one needs to control them. The paper [20] contains more information about caustics for Euclidean billiards. The preceding discussion applies to Birkhoff billiards in a Minkowski plane just as well. The following lemma, besides being useful for the proof of Theorem 6.4, is of interest by itself.

Lemma 6.3. *Let M be a Birkhoff billiard table in a Minkowski plane. Let γ be a caustic corresponding to an invariant circle of the corresponding billiard map. Then γ is contained in the interior of M.*

Proof. Let $0 \le t < 1$ be a parameter on the boundary N of the billiard table. The phase space Φ of the billiard map consists of pairs (t, u) where u is an oriented inward Minkowski unit vector at the point $t \in N$. Let $C \subset \Phi$ be an invariant circle. By Birkhoff's theorem,

$$C = \{(t, u(t)), 0 \le t \le 1\},$$

where $u(t)$ is a Lipschitz function. The billiard map, restricted to C, has the form $\phi(t, u) = (s(t), u(s(t)))$, where the function $s(t)$ is monotonically increasing. Let $t_1 = t + \varepsilon > t$ be

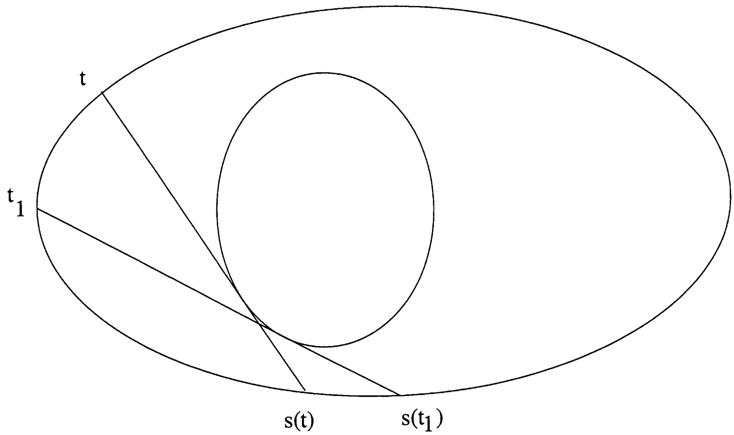

Fig. 7. The billiard map on the invariant circle corresponding to a caustic.

a close point. Then the straight lines, corresponding to $(t, u(t)) \in C$ and $(t_1, u(t_1)) \in C$ intersect in the interior of M — see Fig. 7. Letting $\varepsilon \to \infty$, we obtain the claim. □

Theorem 6.4. *Let M be a Birkhoff billiard table in a Minkowski plane. Let $N = \partial M$ be the boundary. If the Minkowski curvature of N has a zero, then the Minkowski billiard map in M does not have invariant circles.*

Proof. We adapt for the Minkowski geometry the approach of Wojtkowski [43], combined with the argument of Mather [28]. Assume the opposite, and let C be an invariant circle. Let $s \in N$ be a point of curvature zero, and let $(s, v) \in C$ be the corresponding point in the curve. Set $(t, u) = T^{-1}(s, v)$, where T is the billiard map. The light ray corresponding to (t, u) hits N at s, and (s, v) is the reflected ray.

Assume first that C is differentiable at t. Then the rays (t, u) and (s, v) are tangent to the caustic γ, corresponding to C. The two points of tangency, say A and B, form a pair of focusing points that Proposition 6.1 makes a reference to. We apply the Minkowski mirror equation (12). Consider the bilinear form $d^2L(u)$ in the plane. Since the indicatrix I is quadratically convex, $d^2L(u)$ is positive on the tangent line T_uI. On the other hand, L is homogeneous of degree 1 and satisfies Euler's equation $dL(u)(u) = L$. Differentiating again, we obtain $d^2L(u)(u, \cdot) = 0$. Thus the bilinear form $d^2L(u)$ is non-negative, and its kernel is spanned by the vector u.

Hence $d^2L(u)(I', I')$ and $d^2L(v)(I', I')$ are positive. By Lemma 6.3, the points A and B are inside M, thus $a, b > 0$. Since $K = 0$, the left-hand side of Eq. (12) is positive. This is a contradiction.

Suppose now that C is not smooth at t. Since C is Lipschitz, the set of non-smooth points has measure zero. Let (t_n, u_n) be a sequence of smooth points of C, converging to (t, u). Let $A_n, B_n, a_n, b_n, u_n, v_n, K_n$, etc., be the corresponding parameters in (12). The terms $d^2L(u_n)(I'_n, I'_n)$, $d^2L(v_n)(I'_n, I'_n)$ and $[dL(u_n) + dL(v_n)](I''_n)$ converge to finite limits, as

$n \to \infty$. Although a_n and b_n may not have limits, they are positive, and bounded away from infinity, by Lemma 6.3. Since $K_n \to 0$, the left-hand sides of the corresponding equations (12) are positive for n sufficiently large. This is again a contradiction. □

7. Duality for Minkowski billiards

Let U be a vector space of n dimensions, and let $V = U^*$ be the dual of U. Assume that both U and V are equipped with Minkowski metrics (which are *not* necessarily dual to each other). Let $M \subset U$ be the figuratrix of the Minkowski metric in V and $N \subset V$ be the figuratrix of the Minkowski metric in U. We consider M (resp. N) as the boundary of a Minkowski billiard table in U (resp. V). We will establish an orbitwise isomorphism between the two billiard systems.

Let Φ and Ψ be the respective phase spaces, and let $S : \Phi \to \Phi$ and $T : \Psi \to \Psi$ be the corresponding Minkowski billiard maps. We will denote by the same symbol D both Legendre transforms. Throughout this section, Φ (resp. Ψ) is realized as the space of pairs (q, p) (resp. (p, q)) where $q \in M$, $p \in N$ are such that $D(p)$ (resp. $D(q)$) is an inward directed vector with the foot point q (resp. p). That is,

$$\Phi = \{(q, p) \in M \times N | D(p) \cdot D(q) \leq 0\},$$

and

$$\Psi = \{(p, q) \in N \times M | D(q) \cdot D(p) \leq 0\},$$

where $\cdot$ is the pairing between vectors and covectors. The transposition $\tau(q, p) = (p, q)$ is a natural isomorphism between Φ and Ψ.

Theorem 7.1. *A sequence*

$$\ldots, (q_{-2}, p_{-2}), (q_{-1}, p_{-1}), (q_0, p_0), (q_1, p_1), (q_2, p_2), \ldots$$

is an orbit of S if and only if the sequence

$$\ldots (p_{-2}, -q_{-1}), (p_{-1}, -q_0), (p_0, -q_1), (p_1, -q_2), \ldots$$

is an orbit of T.

Proof. In view of the symmetry, it suffices to prove that if the former sequence is a billiard orbit, then so is the latter. We will use $u \sim v$ to denote that the two vectors are proportional with a positive coefficient.

The billiard segment $[q_i q_{i+1}]$ of the phase point $(q_i, p_i) \in \Phi$ belongs to the line, spanned by the vector $D(p_i)$. The billiard reflection takes (q_{i+1}, p_i) to $(q_{i+1}, p_{i+1}) \in \Phi$. We write the respective conditions as

$$q_{i+1} - q_i \sim D(p_i), \tag{14}$$

and

$$p_{i+1} - p_i \sim -D(q_{i+1}). \tag{15}$$

We claim that the phase point $(p_{i-1}, -q_i) \in \Psi$ travels along the ray spanned by the vector $D(q_i)$ to the point $(p_i, -q_i)$, and after the billiard reflection becomes $(p_i, -q_{i+1})$. This is tantamount to the conditions,

$$p_i - p_{i-1} \sim D(-q_i), \quad \text{and} \quad (-q_{i+1}) - (-q_i) = q_i - q_{i+1} \sim D(-p_i).$$

But the former is (15) and the latter is (14). □

By Theorem 7.1, the Euclidean billiard in a centrally symmetric smooth strictly convex table is equivalent to a Minkowski billiard in the Euclidean unit ball.

Remark 7.2. Let Φ^∞ and Ψ^∞ be the spaces of billiard orbits, viewed as bi-infinite sequences in Φ and Ψ, respectively. Theorem 7.1 provides a transformation,

$$\mathcal{F}_{MN} : \Phi^\infty \to \Psi^\infty, \qquad \mathcal{F}_{MN}(q_i, p_i) = (p_i, -q_{i+1})$$

satisfying

$$\mathcal{F}_{MN} \circ S = T \circ \mathcal{F}_{MN}.$$

The composition $\mathcal{F}_{NM} \circ \mathcal{F}_{MN} : \Phi^\infty \to \Phi^\infty$ is given by

$$\begin{aligned} &\ldots, (q_0, p_0), (q_1, p_1), (q_2, p_2), \ldots \\ &\quad \to \ldots, (-q_1, -p_1), (-q_2, -p_2), (-q_3, -p_3), \ldots . \end{aligned}$$

We will apply the above duality to periodic Minkowski billiard orbits. An orbit

$$\{(q_i, p_i), -\infty < i \infty\} \in \Phi^\infty (\text{resp. } \{(p_i, q_i), -\infty < i < \infty\} \in \Psi^\infty)$$

is n-periodic if $q_{i+n} = q_i$, $p_{i+n} = p_i$. We set $\times^n \Phi = \mathcal{C}_n(\Phi)$ and $\times^n \Psi = \mathcal{C}_n(\Psi)$. We view these spaces as the spaces of n-cyclic configurations. For instance,

$$\mathcal{C}_n(\Phi) = \{(q_i, p_i) \in \Phi | q_{i+n} = q_i, p_{i+n} = p_i, -\infty < i < \infty\}.$$

Then $\mathcal{P}_n(M) = \mathcal{C}_n(\Phi) \cap \Phi^\infty$ and $\mathcal{P}_n(N) = \mathcal{C}_n(\Psi) \cap \Psi^\infty$ consist of n-periodic billiard orbits in M and N, respectively.

Proposition 7.3. *Let the notation be as in Theorem* 7.1. *We define the functions* $\mathcal{G}_n^M$ *and* $\mathcal{G}_n^N$ *on* $\mathcal{C}_n(\Phi)$ *and* $\mathcal{C}_n(\Psi)$, *respectively, by*

$$\mathcal{G}_n^M(\vec{q}, \vec{p}) = \sum_{i=1}^{n} (q_{i+1} - q_i) \cdot p_i, \tag{16}$$

and

$$\mathcal{G}_n^N(\vec{p}, \vec{q}) = \sum_{i=1}^{n} (p_{i+1} - p_i) \cdot q_i. \tag{17}$$

Then

1. *The set* $\mathcal{P}_n^M$ (*resp.* $\mathcal{P}_n^N$) *is contained in the set of critical points of the function* $\mathcal{G}_n^M$ (*resp.* $\mathcal{G}_n^N$).
2. *The isomorphism* $\mathcal{F}_{MN} : \mathcal{C}_n(\Phi) \to \mathcal{C}_n(\Psi)$ *sends the function* $\mathcal{G}_n^N$ *into* $\mathcal{G}_n^M$.

Proof. By symmetry, it suffices to prove the first part of claim 1. Consider first the partial derivative $\partial\mathcal{G}_n^M/\partial p_i$. If $q_{i+1}-q_i \neq 0$ then $\partial\mathcal{G}_n^M/\partial p_i = 0$ if and only if the vectors $D(p_i)$ and $q_{i+1} - q_i$ are collinear. In particular, this holds for a periodic billiard orbit $(\vec{q}, \vec{p}) \in \mathcal{C}_n(\Phi)$.

Next, one has the identity

$$\mathcal{G}_n^M(\vec{q}, \vec{p}) = -\sum_{i=1}^{n}(p_i - p_{i-1}) \cdot q_i \tag{18}$$

that follows from (16) by a discrete "integration by parts". This identity implies claim 2 of the proposition.

To finish the proof, consider the partial derivative $\partial\mathcal{G}_n^M/\partial q_i$ in (16). As before, if $p_i - p_{i-1} \neq 0$ then $\partial\mathcal{G}_n^M/\partial q_i = 0$ if and only if the vectors $D(q_i)$ and $p_i - p_{i-1}$ are collinear. This condition is the Finsler billiard reflection law at point q_i, and it holds for a periodic billiard orbit $(\vec{q}, \vec{p}) \in \mathcal{C}_n(\Phi)$. □

Remark 7.4. Pushkar' [32,33] used a function similar to $\mathcal{G}_2^M$ to study the diameters of an immersed submanifold M (e.g., a torus) in Euclidean space.

Remark 7.5. The function $\mathcal{G}_n^M$ has other critical points that do not describe genuine billiard orbits. These "parasite" critical points correspond to the cases when $p_i = p_{i+1}$ or $q_i = q_{i+1}$ for some index i.

The statement below is immediate from Theorem 7.1 and the proof of Proposition 7.3.

Corollary 7.6. *The transformation* $\mathcal{F}_{MN}$ *induces a length-preserving isomorphism between the n-periodic orbits of the Minkowski billiards in M and in N.*

Theorem 7.1 establishes a bijection between Minkowski billiard orbits in "dual tables". Specializing it, we will obtain a duality between Minkowski billiard orbits in the same table. Let $A : V \to V^*$ be an invertible self-adjoint operator. Suppose that $A(M) = N$.

Corollary 7.7. *A sequence*

$$\ldots, (q_{-1}, p_{-1}), (q_0, p_0), (q_1, p_1), \ldots$$

is a billiard orbit in M with respect to the Minkowski metric with the figuratrix N if and only the same holds for the sequence

$$\ldots, (A^{-1}(p_{-1}), -A(q_0)), (A^{-1}(p_0), -A(q_1)), (A^{-1}(p_1), -A(q_2)), \ldots .$$

Proof. The transformation $\mathcal{F}_{MN}$ yields a bijection between Minkowski billiard orbits in M and N. The linear isomorphism

$$A^{-1} \times A : V^* \times V \to V \times V^*$$

induces a diffeomorphism $M \times N \to N \times M$. It remains to show that it sends Minkowski billiard orbits to Minkowski billiard orbits. Since linear maps commute with the Legendre transform, we have $D \circ A = A^{-1} \circ D$. The Minkowski billiard reflection conditions,

$$p_i - p_{i-1} \sim -D(q_i), \qquad q_{i+1} - q_i \sim D(p_i)$$

are linearly invariant. □

Example 7.8. Veselov [39,40] discovered a special case of Corollary 7.7. He called the mapping of Corollary 7.7 "the skew hodograph transformation". In Veselov's case, $V = V^*$ is Euclidean space, $M \subset V$ is the ellipsoid $A^2(q) \cdot q = 1$, and N is the unit sphere $p \cdot p = 1$.

Remark 7.9. Continuous counterparts of Theorem 7.1 and Corollary 7.6 have been recently found by Alvarez — see [4]. Endow $M \subset U$ (resp. $N \subset V$) with the Finsler metric that is the restriction of the Minkowski metric in U (resp. V). One compares the Finsler geodesic flows on M and N. Our description below is somewhat different from that in [4].

The continuous counterpart of Theorem 7.1 is as follows. The unit covector bundle S^*M of M satisfies

$$S^*M = \{(q, p) \in M \times N | D(q) \cdot D(p) = 0\}.$$

The geodesic flow on S^*M is given, in the natural parameter, by

$$q' = D(p), \qquad p' = -\phi D(q),$$

where ϕ is a positive function of (q, p). The transformation $(q, p) \to (p, -q)$ takes S^*M to S^*N, and a geodesic on M into a geodesic on N (but with a non-standard parameterization). If the former geodesic is closed then so is the latter, and both have the same Finsler length.

For the counterpart of Corollary 7.7 note that if $A(M) = N$ then the transformation

$$(q, p) \to (A^{-1}(p), -A(q))$$

sends geodesics on M into geodesics on M.

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